

Homework 8 Solution Key

Math 121A, Fall 2025.

Due: Wednesday, Oct 22, 11:59 PM

Unless otherwise stated, all solutions require clear, well-written justification. You must explicitly credit any collaborators for each problem.

Problem 1. First,

$$\begin{aligned}\frac{1}{e^z - 1} &= \frac{1}{1 + z + z^2/2 + z^3/6 + O(z^4) - 1} \\ &= \frac{1}{z} \left(\frac{1}{1 + z/2 + z^2/6 + O(z^3)} \right) \\ &= \frac{1}{z} (1 - (z/2 + z^2/6 + O(z^3)) - (z/2 + z^2/6 + O(z^3))^2 - O(z^3)) \\ &= \frac{1}{z} (1 - z/2 - z^2(1/6 + 1/4) + O(z^3)) \\ &= \frac{1}{z} - \frac{1}{2} - z(1/6 + 1/4) + O(z^2).\end{aligned}$$

Now for cotangent: in the problem statement we have already computed the first few terms for the Laurent series for $\frac{1}{\sin z} = \frac{1}{z} + \frac{z}{3!} + O(z^3)$, so

$$\begin{aligned}\frac{\cos z}{\sin z} &= (\cos z) \left(\frac{1}{z} + \frac{z}{3!} + O(z^3) \right) \\ &= \left(1 - z^2/2 + O(z^4) \right) \left(\frac{1}{z} + \frac{z}{3!} + O(z^3) \right) \\ &= \frac{1}{z} + z(1/6 - 1/2) + O(z^3).\end{aligned}$$

Problem 2. To find the principal parts of the Laurent expansion, we use the results of problem 2: Find the principal parts of the Laurent expansions about 0 of the following functions, indicating the type of singularity in each case.

(a) For this one, we just square the Laurent expansion of $\frac{1}{\sin z}$:

$$\begin{aligned}\frac{1}{\sin^2(z)} &= \left(\frac{1}{z} + \frac{z}{3!} + O(z^3) \right)^2 \\ &= \frac{1}{z^2} + \frac{2}{6} + O(z^2).\end{aligned}$$

So the principal part is just $\frac{1}{z^2}$ and we have a pole of order 2.

(b) For this one, we use our earlier expansion for $\frac{1}{e^z - 1}$:

$$\begin{aligned}\frac{e^z + 1}{e^z - 1} &= (2 + z + z^2/2 + z^3/3! + O(z^4)) \left(\frac{1}{z} (1 - z/2 - z^2(1/6 + 1/4) + O(z^3)) \right) \\ &= \frac{2}{z} + O(1),\end{aligned}$$

so the principal part of the Laurent expansion is just $2/z$.

Problem 3. (a) $\frac{e^z}{z^2 - 1}, z = 1$. We want expansions valid near $z = 1$, so we massage the expressions to get an exponential series and a geometric series near $z = 1$ (say, with $|z - 1| < 2$):

$$\begin{aligned}\frac{e^z}{z^2 - 1} &= \frac{e \cdot e^{z-1}}{(z-1)(z+1)} = \left(\frac{e}{z-1} \right) \left(\frac{e^{z-1}}{2 + (z-1)} \right) \\ &= \left(\frac{e}{2(z-1)} \right) (e^{z-1}) \left(\frac{1}{1 + \frac{(z-1)}{2}} \right) \\ &= \frac{e}{2(z-1)} \left(1 + (z-1) + \frac{(z-1)^2}{2!} + \dots \right) \left(1 - \frac{z-1}{2} + \left(\frac{z-1}{2} \right)^2 + \dots \right) \\ &= \frac{e}{2(z-1)} \left(1 + \frac{z-1}{2} + (z-1)^2 \left(\frac{1}{2!} + \frac{1}{4} - \frac{1}{2} \right) + \dots \right) \\ &= \frac{e}{2(z-1)} + \frac{e}{4} + \frac{e}{8}(z-1) + \dots\end{aligned}$$

So $\text{Res}_1 f = e/2$.

(b) $\sin \frac{1}{z}, z = 0$. In this case the Laurent expansion is quite immediate, since $\sin z$ is entire:

$$\sin \frac{1}{z} = \frac{1}{z} - \frac{1}{3!} \frac{1}{z^3} + \frac{1}{5!} \frac{1}{z^5} + \dots$$

So $\text{Res}_0 f = 1$.

Problem 4. (a) Problem 14.4.3. The function $f(z) = \frac{1}{z(z-1)(z-2)}$ has singularities at $z = 0, 1, 2$. We care about the Laurent series around $z = 0$, so we pull out $1/z$ and we do a partial fraction expansion on the other two:

$$\frac{1}{z(z-1)(z-2)} = \frac{1}{z} \left(\frac{1}{z-2} - \frac{1}{z-1} \right).$$

As with the example in class, we have three regimes, annuli whose boundaries are the singularities: ($0 < |z| < 1$):

$$f(z) = \frac{1}{z} \left(-\frac{1}{2} \frac{1}{(1-z/2)} + \frac{1}{(1-z)} \right).$$

The first term converges for $|z| < 2$, and the second for $|z| < 1$. So we are good to go:

$$\begin{aligned}f(z) &= \frac{1}{z} \left(-\frac{1}{2} \left(1 + z/2 + (z/2)^2 + \dots \right) + (1 + z + z^2 + \dots) \right) \\ &= \frac{1}{2z} + \frac{3}{4} + \frac{7}{8}z + \dots\end{aligned}$$

So the residue is $\text{Res}_0 f = \frac{1}{2}$.

($1 < |z| < 2$): Now, the first expansion is still fine since $|z| < 2$, but the second we modify by $|z| > 1$, or $|\frac{1}{z}| < 1$ to get a convergent expansion:

$$f(z) = \frac{1}{z} \left(-\frac{1}{2} \frac{1}{(1 - z/2)} - \frac{1}{z} \left(\frac{1}{(1 - 1/z)} \right) \right).$$

Soldiering on, we have

$$\begin{aligned} f(z) &= \frac{1}{z} \left(-\frac{1}{2} \left(1 + z/2 + (z/2)^2 + \dots \right) - \frac{1}{z} \left(1 + 1/z + 1/z^2 + \dots \right) \right) \\ &= \frac{1}{z} \left(-\frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} + \dots - \frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} \dots \right). \end{aligned}$$

($|z| > 2$): Finally, we have entered the regime where $|z| > 2$, or $|\frac{1}{z}| < 2$. We adjust our expansions to get convergent geometric series once again:

$$f(z) = \frac{1}{z} \left(\frac{1}{z} \frac{1}{(1 - 2/z)} - \frac{1}{z} \left(\frac{1}{(1 - 1/z)} \right) \right).$$

So we arrive at

$$\begin{aligned} f(z) &= \frac{1}{z^2} \left(1 + 2/z + (2/z)^2 + \dots - 1 - 1/z - 1/z^2 - \dots \right) \\ &= \frac{1}{z^2} \left(1/z + 3/z^2 + \dots \right). \end{aligned}$$

- (b) Problem 14.4.7. The function $f(z) = \frac{2-z}{1-z^2} = \frac{2-z}{(1-z)(1+z)}$ has simple poles at ± 1 . Let's do a partial fraction expansion to find

$$f(z) = -\frac{1/2}{1-z} + \frac{3/2}{1+z}.$$

Now, we only have two regimes if we wish to expand at the origin: $|z| < 1$ and $|z| > 1$.

($|z| < 1$) In this regime, both geometric series converge, so we have

$$\begin{aligned} f(z) &= -\frac{1}{2} \left(1 + z + z^2 + \dots \right) + \frac{3}{2} \left(1 - z + z^2 + \dots \right) \\ &= 1 - 2z + z^2 + \dots \end{aligned}$$

Thus the residue is $\text{Res}_0 f = 0$ (which isn't surprising, there is no pole there!).

($|z| > 1$) In this regime, we adjust to $|\frac{1}{z}| < 1$ to find convergent geometric series:

$$f(z) = \frac{1}{2z} \frac{1}{1 - 1/z} + \frac{3}{2z} \frac{1}{1 + 1/z},$$

so we have

$$f(z) = \frac{1}{2z} \left(1 + 1/z + 1/z^2 + \dots \right) + \frac{3}{2z} \left(1 - 1/z + 1/z^2 + \dots \right).$$

Problem 5. Finding residues.

- (a) $f(z) = \frac{e^z - 1}{\sin z}$, $z_0 = 0$. We start by guessing that f has a simple pole at 0 (since $\sin(z) = z + O(z^3)$ near 0). So, we calculate

$$\lim_{z \rightarrow 0} (z - 0)f = \lim_{z \rightarrow 0} \frac{z(e^z - 1)}{\sin z} = \lim_{z \rightarrow 0} \frac{e^z + ze^z - 1}{\cos z} = 0.$$

Thus, $\text{Res}_0 f = 0$. In hindsight, this was rather obvious, since

$$\frac{e^z - 1}{\sin z} = \frac{1 + z + z^2/2 + z^3/6 + O(z^4) - 1}{z - z^3/6 + O(z^4)} = \frac{z + z^2/2 + z^3/6 + O(z^4)}{z - z^3/6 + O(z^4)} = \frac{1 + z/2 + z^2/6 + O(z^3)}{1 - z^2/6 + O(z^3)},$$

and so this quantity does not blow up near $z = 0$.

- (b) $\frac{e^z}{(z^2 - 1)^2}$, $z_0 = 1$. We suspect a pole of order 2, since the denominator factors into $(z + 1)^2(z - 1)^2$ and the numerator has no zeros. So:

$$\frac{d}{dz} \left((z - 1)^2 \frac{e^z}{(z^2 - 1)^2} \right) \Big|_{z=1} = \frac{d}{dz} \left(\frac{e^z}{(z + 1)^2} \right) \Big|_{z=1} = 0.$$

So $\text{Res}_1 \frac{e^z}{(z^2 - 1)^2} = 0$.

- (c) $\left(\frac{\cos z - 1}{z}\right)^2$, $z_0 = 0$. We suspect a pole of order 2, so we again compute

$$\frac{d}{dz} \left(z^2 \left(\frac{\cos z - 1}{z} \right)^2 \right) \Big|_{z=0} = \frac{d}{dz} \left((\cos z - 1)^2 \right) \Big|_{z=0} = 0.$$

So $\text{Res}_0 \left(\frac{\cos z - 1}{z}\right)^2 = 0$. This was another in retrospect unsurprising one, since

$$\left(\frac{\cos z - 1}{z}\right)^2 = \left(\frac{1 - z^2/2 + O(z^4) - 1}{z}\right)^2 = \left(-z/2 + O(z^3)\right)^2,$$

i.e. there are no poles in this function.

- (d) $\frac{z+2}{z^2-2z}$, $z_0 = 0$. Here we have two simple poles, one of which is 0, so we can compute

$$\lim_{z \rightarrow 0} z \frac{z+2}{z^2-2z} = \lim_{z \rightarrow 0} \frac{z+2}{z-2} = -1,$$

so $\text{Res}_{-1} \frac{z+2}{z^2-2z} = -1$.

Problem 6. Give examples of functions f_1, f_2 with the following properties:

- (a) Here's one:

$$f_1(z) = \frac{1}{z} + (z - i) + \exp\left(\frac{1}{z - 1}\right).$$

- (b) Here's one:

$$f_2(z) = \begin{cases} 10 & z = 0 \\ \frac{1}{(z-1)^6} + \exp\left(\frac{1}{z-i}\right) & z \neq 0. \end{cases}$$

Problem 7. Locate and classify (as removable, essential, or poles of a specified order) the singularities of the following functions. We will need a lemma.

Lemma 0.1. *The only solutions to $\sin z = 0$ (resp. $\cos z = 0$) are $z = \pi k$, $k \in \mathbb{Z}$ (resp. $z = \pi k + \pi/2$, $k \in \mathbb{Z}$).*

Proof. Observe that if $z = x + iy$,

$$|\sin(x + iy)| = \left| \frac{e^{ix}e^{-y} - e^{-ix}e^y}{2i} \right| \geq \left| \frac{|e^{ix}e^{-y}| - |e^{-ix}e^y|}{2} \right| = \left| \frac{e^{-y} - e^y}{2} \right|.$$

But evidently the right hand side is zero only when $y = 0$. Thus all zeros of $\sin z$ are on the real line, where we know what they are from calculus. $\square$

- (a) $\frac{1}{(\pi+z)\sin(z)}$. By the lemma, we have poles for all πk , $k \in \mathbb{Z}$. At $z = -\pi$, we have a pole of order 2, since

$$\lim_{z \rightarrow -\pi} (\pi + z) \frac{1}{(\pi + z)\sin(z)} = \infty, \quad \text{but} \quad \lim_{z \rightarrow -\pi} (\pi + z)^2 \frac{1}{(\pi + z)\sin(z)} = -1.$$

For all other πk , we have simple poles, since if $k \neq -1$ then

$$\lim_{z \rightarrow k\pi} (z - k\pi) \frac{1}{(z + \pi)\sin(z)} = \frac{1}{(k\pi + \pi)(\cos(k\pi))} = \begin{cases} \frac{1}{(k+1)\pi} & k \text{ even} \\ \frac{-1}{(k+1)\pi} & k \text{ odd.} \end{cases}$$

There are no other singularities.

- (b) $\frac{1}{(z^2+z+1)^3}$. We have to factor the denominator. Using $z^3 - 1 = (z - 1)(z^2 + z + 1)$, we realize that the roots are the two nontrivial third roots of unity (I also don't mind you asking wolfram):

$$(z^2 + z + 1) = (z - e^{2\pi i/3})(z - e^{4\pi i/3}).$$

It is straightforward to verify that both of these roots give poles of order 3 in the original function, since

$$\frac{1}{(z^2 + z + 1)^3} = \frac{1}{(z - e^{2\pi i/3})^3(z - e^{4\pi i/3})^3}.$$

- (c) $\sin(1 - 1/z)$. This one has an essential singularity at 0, which can be seen in a few ways. The first is to plug it into its series and see the Laurent expansion:

$$\sin(1 - 1/z) = (1 - 1/z) - \frac{(1 - 1/z)^3}{3!} + \dots \pm \frac{(1 - 1/z)^{2n+1}}{(2n+1)!} + \dots,$$

which evidently has infinitely many nonzero coefficients of $\frac{1}{z^k}$ after multiplying out, so it must be an essential singularity.

[Here's an alternative argument which avoids Laurent series.](#) To see that 0 is an essential singularity, we show that for any $m > 0$, the following limit

$$\lim_{z \rightarrow 0} z^m \sin(1 - 1/z)$$

does not exist, and so this singularity is neither removable nor a pole. To see this, it suffices to check this on odd m and then observe that by plugging in $1 - 1/z$ into the power series defining $\sin w$,

$$\begin{aligned}\lim_{z \rightarrow 0} z^m \sin(1 - 1/z) &= \lim_{z \rightarrow 0} z^m \left((1 - 1/z) - \frac{1}{3!}(1 - 1/z)^3 + \cdots \pm \frac{1}{m!}(1 - 1/z)^m \pm \frac{1}{(m+2)!}(1 - 1/z)^{m+2} + \cdots \right) \\ &= \lim_{z \rightarrow 0} 0 \pm \frac{1}{m!}(z - 1)^m \pm \frac{1}{(m+2)!}(z - 1)^m(1 - 1/z)^2 + \cdots,\end{aligned}$$

which is unbounded in every neighborhood of 0, so this limit does not exist. One could instead show that 0 is not a removable singularity, and the restriction to the real line $\sin(1 - 1/x)$ is bounded and so cannot be a pole. This leaves essential singularity.

- (d) $\csc(\sin(z))$, which we may re-write as $\frac{1}{\sin(\sin(z))}$. There is one obvious set of poles, from $z = \pi k$, $k \in \mathbb{Z}$. These can all be seen to be simple, since

$$\lim_{z \rightarrow \pi k} (z - \pi k) \frac{1}{\sin(\sin(z))} = \lim_{z \rightarrow \pi k} \frac{1}{\cos(\sin(z)) \cos(z)} = \frac{1}{\cos(\pi k)} = \pm 1.$$

We then ask whether there are other possible zeros, which by the earlier lemma, may only appear if $\sin(z) = \pi k$, so we look for solutions to this. Notice that this is purely real, so it is easiest to start by looking for solutions to

$$0 = \operatorname{Im}(\sin(x + iy)) = \cos(x) \sinh(y).$$

$\sinh(y) = 0$ if and only if $y = 0$, and $\cos(x) = 0$ if and only if $x = \pi/2 + \pi n$. If $y = 0$, then z is purely real and so $|\sin(z)| \leq 1$ and so there are no solutions to $\sin(z) = \pi k$ with $k \neq 0$. This forces $x = \pi/2 + \pi n$. Then, we look for solutions to

$$\pi k = \operatorname{Re}(\sin(\pi/2 + \pi n + iy)) = \sin(\pi/2 + \pi n) \cosh(y) = \pm \cosh(y).$$

A quick glance at a graph of $\cosh(y)$ reveals that for each $k \neq 0$, there are exactly two solutions to this equation, call them y_k^+ and y_k^- . So, we also have a set of poles coming from the points $(\pi/2 + \pi n, y_k^+)$ and $(\pi/2 + \pi n, y_k^-)$, where $n \in \mathbb{Z}$ and $k \in \mathbb{Z}$. It is immediate that these poles are simple, because the zeros of sine are all simple.

Problem 8. Observe that $f(z) \neq 0$ for all z . Thus the function $\frac{1}{f(z)}$ is entire, and $\left| \frac{1}{f(z)} \right| \leq 1$. Thus Liouville's theorem guarantees $\frac{1}{f(z)}$ is constant, say $\frac{1}{f(z)} = c$, so $f(z) = \frac{1}{c}$ is also constant.

Problem 9. We begin by observing that this implies that the function $\left| \frac{f(z)}{z} \right| \leq M$, so this function is bounded and holomorphic on all $z \neq 0$. To handle the case where $z = 0$, we take a Taylor expansion near 0:

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

But since $|f(0)| \leq M|0| = 0$, this forces $f(0) = 0$ so $a_0 = 0$ and we have

$$f(z) = a_1 z + a_2 z^2 + \cdots = z(a_1 + a_2 z + \cdots),$$

so in particular, we have that $\frac{f(z)}{z}$ is given by a convergent power series and so it is holomorphic at 0, taking on the value 0 there. So, the function $\frac{f(z)}{z}$ is entire and bounded, and thus by Liouville's theorem, it is constant, say $\frac{f(z)}{z} = a$ for some $a \in \mathbb{C}$. This means that $f(z) = az$, which is what we wanted to show.

Problem 10. (a) By Cauchy's integral formula applied to a circle γ of radius r centered at z_0 , we have

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z_0} d\zeta = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(re^{i\theta})}{(z_0 + re^{i\theta}) - z_0} ire^{i\theta} d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta.$$

(b) Let $u : \mathbb{C} \rightarrow \mathbb{R}$ be given by $u(z) = \operatorname{Re}(f(z))$. Notice that $d\theta$ is a differential of a real parameter, so we may freely use real-linearity of the function Re to write

$$\operatorname{Re}(f(z_0)) = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re}(f(z_0 + re^{i\theta})) d\theta,$$

which is to say

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta.$$